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Twisted Goldstone models

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Abstract. Symmetry breaking for twisted scalar fields in non-simply connected space–times is studied. A general procedure is described whereby an approximation to the ground state in the broken phase may be found when the length scales associated with the non-trivial topology are close to their critical values at which the phase transition occurs. Application to twisted fields in $S^1\times\mathbb{R}^3$ and $\mathbb{R}^1\times\mathbb{RP}^3$ is given.

1. Introduction

There is currently much interest in studying symmetry breaking in space–times which are different from Minkowski space–time, the ultimate aim being to investigate the cosmological consequences in the early universe. In addition to the effects of space–time curvature, it is found that a non-trivial topology can also have an effect upon vacuum stability. There can be critical length scales (or curvatures) introduced in either case at which phase transitions occur. (See Avis and Isham 1978, Banach 1981, Critchley and Dowker 1982, Denardo *et al* 1981, Denardo and Spallucci 1980, 1981a, b, c, 1982, Fawcett and Whiting 1982, Ford 1980, Ford and Toms 1982, Gibbons 1978, Kennedy 1981, Shore 1980, Toms 1980b, c, 1982a, b, Unwin 1982 for some of the features which can arise.)

Another characteristic of quantum field theory in topologically non-trivial space–times is the possible existence of inequivalent types of fields with the same spin (Banach and Dowker 1979, Dowker and Banach 1978, Isham 1978a, b). The number of inequivalent real scalar fields—defined to be cross-sections of real line bundles over a space–time M—is given by the order of the cohomology group $H^1(M;Z_2)\approx \mathrm{Hom}(\pi_1(M),Z_2)$ (Isham 1978a). It therefore follows that whenever the space–time is non-simply connected, twisted real scalar fields occur which are cross-sections of non-product bundles. Typically, these twisted fields satisfy antiperiodic boundary conditions so that the only constant such field is the zero field. As a consequence symmetry breaking is altered even at the classical level (Avis and Isham 1978) since any non-zero ground state is necessarily position dependent. Finding the vacuum state when symmetry breaking occurs involves solving nonlinear differential equations, a very difficult task in general.

The effective potential method (Coleman and Weinberg 1973, Weinberg 1973), which is normally used to study symmetry breaking beyond the tree-level, may not be used for twisted fields (Toms 1980b) since a crucial step in obtaining it involves

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setting all fields to constants. There have been three main proposals for studying symmetry breaking for twisted fields beyond the classical level (Banach 1981, Ford 1980, Toms 1982). Ford's method includes quantum corrections to the classical field equation and then identifies modes which grow with time as unstable. It gives the same results as the method of Toms (1982a) which involves studying the eigenvalues of the second functional derivative of the effective action (see § 2 below). Banach's (1981) approach defines a generalised effective potential by

$$V(A)=\frac{1}{\mathrm{vol}(M)}\Gamma[A\varphi_0(x)] \tag{1}$$

where A is a constant, vol(M) is the space–time volume, Γ is the effective action, and $\varphi_0(x)$ is the eigenfunction with the lowest eigenvalue of the space–time derivative part of the classical action. $V(A)$ is then analysed like an ordinary potential.

The method described in the present paper is related to that of Banach (1981) although our approach is somewhat different. We show how it is possible to find approximate solutions to the nonlinear field equations when the characteristic length scales in the problem are close to the critical values at which the phase transition occurs. We believe that our method clarifies Banach's (1981) paper and explains clearly why he found good results as the critical length was approached. A systematic procedure is described for going beyond this lowest-order approximation. Application to twisted scalar fields in $S^1\times\mathbb{R}^3$ and $\mathbb{R}^1\times\mathbb{RP}^3$ is given.

2. The general method

Consider a real scalar field whose classical action functional is

$$I[\varphi]=\int \mathrm{d}v_x\left[-\tfrac{1}{2}\varphi\Box\varphi+\tfrac{1}{2}\xi R\varphi^2+\frac{\lambda}{4}(\varphi^2-v^2)^2\right] \tag{2}$$

where $\mathrm{d}v_x=(g(x))^{1/2}\,\mathrm{d}^4x$ is the invariant volume element, R is the scalar curvature, and ξ, v, $\lambda>0$ are constants. We choose for convenience to work in a Riemannian space–time rather than a Lorentzian one, although this is not strictly necessary. The potential term in (2) is taken to be of the Goldstone (1961), or double-well type.

The classical (or tree-level) ground states are the solutions $\varphi_c(x)$ to

$$\delta I[\varphi]/\delta\varphi(x)|_{\varphi_c}=0 \tag{3}$$

which minimise $I[\varphi]$. From (2) the equation of motion is

$$-\Box\varphi_c(x)+\xi R(x)\varphi_c(x)+\lambda\varphi_c(x)[\varphi_c^2(x)-v^2]=0. \tag{4}$$

One way to study the stability of solutions to (4) is to examine the spectrum of the differential operator

$$\delta^2 I[\varphi]/\delta\varphi(x')\delta\varphi(x)|_{\varphi_c}=[-\Box_x+\xi R(x)+\lambda(3\varphi_c^2(x)-v^2)]\delta(x,x') \tag{5}$$

where $\delta(x,x')$ is the biscalar Dirac distribution. We are therefore interested in the eigenvalues λ_N defined by

$$[-\Box_x+\xi R(x)+\lambda(3\varphi_c^2(x)-v^2)]\Psi_N(x)=\lambda_N\Psi_N(x). \tag{6}$$

If λ_0 denotes the smallest eigenvalue of (6), then $\varphi_c(x)$ is an unstable ground state if $\lambda_0<0$, and is locally stable otherwise.

In a flat, or more generally in a homogeneous, stationary space–time, the ground state will be expected to be a constant solution to (4). It is unstable if

$$\xi R + \lambda(3\varphi_c^2 - v^2) < 0. \tag{7}$$

This has relied on the fact that both the curvature and ground state are constant. If either of these conditions is not met, then finding solutions to (4), as well as analysing the stability in (6), becomes much more difficult.

One important case where a more detailed analysis is necessary occurs when twisted scalar fields are considered. For the remainder of this paper we restrict our attention to space–times with a constant scalar curvature. If we examine the stability of $\varphi_c(x) = 0$, which is seen to be a solution to (4), it is seen from (6) to be unstable if

$$l_0^2 + \xi R - \lambda v^2 < 0 \tag{8}$$

where l_0^2 is the lowest eigenvalue of the Laplacian $-\Box_x$ which is necessarily positive. Even in flat space–time ($R = 0$), the fact that $l_0^2 > 0$ may stabilise a theory which would naively be thought of as unstable from the shape of the potential in (2). Both l_0^2 and R will be functions of the length scales which characterise the geometry. Thus, if the quantity on the left-hand side of (8) can assume either sign as these lengths are varied, there will be critical values of the length scales at which $\varphi_c(x) = 0$ becomes unstable. Finding the ground state when $\varphi_c(x) = 0$ is unstable, requires solving (4).

We now wish to include one-loop quantum corrections to this classical analysis. One way to do it, following Toms (1982a), is to evaluate the one-loop effective action using the background field method (De Witt 1965). This leads to, keeping $\hbar$ as a loop-counting parameter,

$$\Gamma[\varphi] = I[\varphi] - \tfrac{1}{2}\hbar \ln \det \Delta_F - \hbar \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \operatorname{Tr}[(\Delta_F \Phi)^n] + O(\hbar^2). \tag{9}$$

Here Δ_F is the Feynman propagator defined by

$$(-\Box_x + \xi R)\Delta_F(x, x') = \delta(x, x') \tag{10}$$

and a shorthand functional notation is adopted in (9) whereby

$$\operatorname{Tr}(\Delta_F \Phi) = \lambda \int dv_x\, \Delta_F(x, x)(3\varphi^2(x) - v^2). \tag{11}$$

We now make the assumption that the dominant term in (9) is the $n = 1$ term. In the examples dealt with below this can be guaranteed by taking the appropriate length scales to be small. This is analogous to the high-temperature expansion used in finite temperature field theory (Dolan and Jackiw 1974, Weinberg 1974). With this approximation in mind, we may write the one-loop effective action as

$$\Gamma[\varphi] = \int dv_x \left\{ -\tfrac{1}{2}\varphi \Box \varphi + \tfrac{1}{2}\xi R \varphi^2 + \frac{\lambda}{4}[\varphi^2 - (v^2 - 3\hbar\Delta_F)]^2 \right\} \tag{12}$$

where terms which are independent of the field have been dropped as they are irrelevant for symmetry breaking. Because attention is restricted here to homogeneous space–times, $\Delta_F(x, x)$ will be a constant, which we now call Δ_F, and acts just like a mass term. ($\Delta_F(x, x)$ is divergent; however, we suppose that the usual renormalisation procedure has been carried out with all coupling constants in (12) now taking their renormalised values. (For details concerning this see Toms (1982a).)

The vacuum state at the quantum level is the solution φ_q to

$$\delta\Gamma[\varphi]/\delta\varphi(x)|_{\varphi_q}=0 \tag{13}$$

which minimises Γ. From (12), it is clear that this is equivalent to the classical problem provided that we replace v^2 with $\bar{v}^2$ where

$$\bar{v}^2=v^2-3\hbar\Delta_{\mathrm{F}}. \tag{14}$$

The ground state is then a solution to (compare (4))

$$-\Box\varphi_q+\xi R\varphi_q+\lambda\varphi_q(\varphi_q^2-\bar{v}^2)=0 \tag{15}$$

for which the eigenvalues λ_N in (compare (6))

$$[-\Box_x+\xi R+\lambda(3\varphi_q^2-\bar{v}^2)]\Psi_N(x)=\lambda_N\Psi_N(x) \tag{16}$$

are all non-negative. Critical values of the length scales are defined as solutions to

$$l_0^2+\xi R-\lambda\bar{v}^2=0 \tag{17}$$

where again l_0^2 is the lowest eigenvalue of $-\Box_x$. $\varphi_q=0$ is a solution to (15) which is unstable if the left-hand side of (17) is negative. From (14), (17), it is seen that quantum corrections can alter the critical lengths from their classical values.

Assume that there is only one length scale L in the problem. The ground state when $\varphi_q=0$ is unstable requires an exact solution to (15), which as we have remarked previously, will be very difficult to find in general. Consider the case when L is very close to its critical value L_c and write

$$L=(1+\varepsilon)L_c \tag{18}$$

where $0\leqslant\varepsilon\ll 1$ is a dimensionless parameter. Take all coordinates to be dimensionless and use (18) to obtain

$$R=R_c+\sum_{n=1}^{\infty}\varepsilon^n R_n, \tag{19a}$$

$$\Box=\Box_c+\sum_{n=1}^{\infty}\varepsilon^n\Box_n, \tag{19b}$$

$$\bar{v}^2=\bar{v}_c^2+\sum_{n=1}^{\infty}\varepsilon^n\bar{v}_n. \tag{19c}$$

A subscript c denotes that L has been set equal to L_c in the indicated quantity. From the argument presented in the appendix we may also write the solution to (15) as

$$\varphi_q(x)=\varepsilon^{1/2}[\varphi_0(x)+\varepsilon\varphi_1(x)+\mathrm{O}(\varepsilon^2)] \tag{20}$$

where φ_0 and φ_1 are independent of ε. By substituting (19*a*, *b*, *c*), (20) into (15) and equating coefficients of equal powers of ε to zero, we are led to an infinite set of coupled differential equations, the first two of which are

$$-\Box_c\varphi_0+\xi R_c\varphi_0-\lambda\bar{v}_c^2\varphi_0=0, \tag{21}$$

$$-\Box_c\varphi_1+\xi R_c\varphi_1-\lambda\bar{v}_c^2\varphi_1=\Box_1\varphi_0-\xi R_1\varphi_0-\lambda\varphi_0^3+\lambda\bar{v}_1\varphi_0. \tag{22}$$

Except for (21), all of the resulting equations are inhomogeneous. Their structure is such that the equation for φ_n is linear in φ_n with the inhomogeneous term involving $\varphi_0,\ldots,\varphi_{n-1}$; thus, they may be solved iteratively beginning with (21), a task which involves only linear differential equations in contrast to the original nonlinear differential equation in (15). From (21), note that φ_0 is an eigenfunction of $-\Box_c$ with the eigenvalue $\lambda\bar{v}_c^2 - \xi R_c$. By the definition of the critical length in (17) this eigenvalue is seen to be just l_0^2; thus, φ_0 is an eigenfunction of $-\Box_c$ which has the lowest eigenvalue. This is just the object used by Banach (1981) in his generalised effective potential. We believe that our method shows clearly why this object occurs as well as the limitation of using it; namely, it is only the lowest-order approximation to a series expansion about the critical length. As L increases beyond L_c, higher and higher order terms such as those indicated in (20), (22) become increasingly important.

Consider first of all the lowest-order contribution to φ_q in which only (21) is retained. Since this is a homogeneous differential equation the overall scale of the solution is not fixed. If $\tilde{\varphi}_0(x)$ represents any solution to (21), then

$$\varphi_0(x) = \hat{\varphi}\tilde{\varphi}_0(x) \tag{23}$$

is also a solution, where $\hat{\varphi}$ is any constant which is independent of ε. (There will in general be more than one linearly independent solution to (21) with the same lowest eigenvalue of $-\Box_c$; however, these different solutions will be related by an action of the isometry group of the space–time so that we may choose any one of them.) The coefficient $\hat{\varphi}$ in (23) is to be fixed by demanding that the effective action be minimised to lowest order in ε. This part of the calculation is akin to the variational method in ordinary quantum mechanics where a trial wavefunction containing as yet unspecified parameters is used to find the energy which is then minimised with respect to the parameters.

From (12), using definition (14), expansions (19*a*, *b*, *c*), (20), and the equation of motion (21) for φ_0, we have

$$\Gamma[\varphi_q] = \Gamma_0 + \tfrac{1}{2}A\hat{\varphi}^2 + \tfrac{1}{4}B\hat{\varphi}^4 \tag{24}$$

where Γ_0 is a constant,

$$A = \varepsilon^2 \int \mathrm{d}v_x[-\tilde{\varphi}_0\Box_1\tilde{\varphi}_0 + \xi R_1\tilde{\varphi}_0^2 - \lambda\bar{v}_1\tilde{\varphi}_0^2] + \mathrm{O}(\varepsilon^3), \tag{25}$$

$$B = \varepsilon^2 \int \mathrm{d}v_x\lambda\tilde{\varphi}_0^4 + \mathrm{O}(\varepsilon^3). \tag{26}$$

Expression (24), which may be recognised as Banach's (1981) generalised effective potential, is now minimised with respect to $\hat{\varphi}$. Setting $(\partial/\partial\hat{\varphi})\Gamma[\varphi_q] = 0$ leads to

$$(A + B\hat{\varphi}^2)\hat{\varphi} = 0. \tag{27}$$

Thus we have $\hat{\varphi} = 0$ or else

$$\hat{\varphi}^2 = -A/B. \tag{28}$$

Note that

$$\partial^2\Gamma[\varphi_q]/\partial\hat{\varphi}^2|_{\hat{\varphi}=0} = A, \tag{29a}$$

$$\left.\frac{\partial^2\Gamma[\varphi_q]}{\partial\hat{\varphi}^2}\right|_{\hat{\varphi}^2=-A/B} = -2A, \tag{29b}$$

which shows that the solution for $\hat{\varphi}$ given in (28) makes Γ a minimum if $A<0$. (B is seen from (26) to be positive, so that $A<0$ is also necessary in order that (28) yield a real solution for $\hat{\varphi}$). The lowest-order contribution to $\varphi_q(x)$ is therefore

$$\varphi_q(x)=\varepsilon^{1/2}[(-A/B)^{1/2}\tilde{\varphi}_0(x)+\mathrm{O}(\varepsilon)] \tag{30}$$

(either sign may be chosen here). This result is seen to be independent of how $\tilde{\varphi}_0$ is normalised.

At the next order in ε we must solve (22) where from (23), (28),

$$\varphi_0(x)=(-A/B)^{1/2}\tilde{\varphi}_0(x) \tag{31}$$

appears in the inhomogeneous term. The general solution may be written as

$$\varphi_1(x)=\varphi_{1\mathrm{h}}(x)+\psi_1(x) \tag{32}$$

where $\varphi_{1\mathrm{h}}(x)$ is the solution to the homogeneous equation (21), and $\psi_1(x)$ is any particular solution to (22). Because the equation satisfied by ψ_1 is inhomogeneous, its overall scale is fixed; however $\varphi_{1\mathrm{h}}(x)$ is not so determined. Without loss of generality we may choose $\varphi_{1\mathrm{h}}(x)$ to be proportional to $\tilde{\varphi}_0(x)$ with the constant of proportionality once again fixed by the requirement that (32) minimises Γ to lowest order in ε. We therefore write

$$\varphi_q(x)=\varepsilon^{1/2}[(-A/B)^{1/2}\tilde{\varphi}_0(x)+\varepsilon(\hat{\varphi}\tilde{\varphi}_0(x)+\psi_1(x))+\mathrm{O}(\varepsilon^2)]. \tag{33}$$

From (12), (14), (33)

$$(\partial/\partial\hat{\varphi})\Gamma[\varphi_q]=\varepsilon^{3/2}\int \mathrm{d}v_x\,\tilde{\varphi}_0(-\Box\varphi_q+\xi R\varphi_q-\lambda\bar{v}^2\varphi_q+\lambda\varphi_q^3)+\mathrm{O}(\varepsilon^5), \tag{34}$$

$$(\partial^2/\partial\hat{\varphi}^2)\Gamma[\varphi_q]=\varepsilon^3\int \mathrm{d}v_x\,\tilde{\varphi}_0(-\Box\tilde{\varphi}_0+\xi R\tilde{\varphi}_0-\lambda\bar{v}^2\tilde{\varphi}_0+3\lambda\varphi_q^2\tilde{\varphi}_0)+\mathrm{O}(\varepsilon^5). \tag{35}$$

These two results may be simplified by using the fact that $\tilde{\varphi}_0$ and ψ_1 are solutions to (21), (22) respectively. Using expansions (19*a*, *b*, *c*) leads to

$$(\partial/\partial\hat{\varphi})\Gamma[\varphi_q]=\varepsilon^4(C+D\hat{\varphi})+\mathrm{O}(\varepsilon^5), \tag{36}$$

$$(\partial^2/\partial\hat{\varphi}^2)\Gamma[\varphi_q]=\varepsilon^4 D+\mathrm{O}(\varepsilon^5), \tag{37}$$

where

$$C=\int \mathrm{d}v_x\left\{\left(-\frac{A}{B}\right)^{1/2}\tilde{\varphi}_0(-\Box_2+\xi R_2-\lambda\bar{v}_2)\tilde{\varphi}_0+\tilde{\varphi}_0\left[-\Box_1+\xi R_1-\lambda\bar{v}_1+3\lambda\left(-\frac{A}{B}\right)\tilde{\varphi}_0^2\right]\psi_1\right\}, \tag{38}$$

$$D=\int \mathrm{d}v_x\,\tilde{\varphi}_0\left[-\Box_1+\xi R_1-\lambda\bar{v}_1+3\lambda\left(-\frac{A}{B}\right)\tilde{\varphi}_0^2\right]\tilde{\varphi}_0. \tag{39}$$

From (36) we find

$$\hat{\varphi}=-C/D \tag{40}$$

which from (47) will minimise the effective action to lowest order in ε provided that $D>0$. We therefore conclude that the first two terms in the expansion of the ground state are

$$\varphi_q(x)=\varepsilon^{1/2}\{(-A/B)^{1/2}\tilde{\varphi}_0(x)+\varepsilon[-(C/D)\tilde{\varphi}_0(x)+\psi_1(x)]+\mathrm{O}(\varepsilon^2)\} \tag{41}$$

where $\tilde{\varphi}_0$ and ψ_1 are any solutions to (21), (22) respectively, and the constants A, B,

C, D are given in (25), (26), (38), (39). It is clear how this method may be extended to higher orders.

3. Some examples

3.1. $S^1 \times \mathbb{R}^3$

Consider flat space–time with one of the spatial coordinates periodically identified to give it a topology of $S^1 \times \mathbb{R}^3$. The line element is chosen to be

$$ds^2 = dt^2 + a^2\, d\theta^2 + dy^2 + dz^2 \tag{42}$$

where $0 \leq \theta \leq 2\pi$, $-\infty < t$, y, $z < +\infty$. The constant a represents the radius of the circle. Since $\pi_1(S^1 \times \mathbb{R}^3) \approx Z$, it is possible to have a twisted real scalar field which satisfies the antiperiodic boundary condition $\varphi(t, \theta + 2\pi, y, z) = -\varphi(t, \theta, y, z)$. The exact solution to the classical equation of motion (4) is known in this case (Avis and Isham 1978). The solution to (15) easily follows, enabling us to test our method against a known solution.

An expression for the coincidence limit of the free Feynman propagator defined in (10) may be found in Birrell and Ford (1980) or Toms (1980a). It is

$$\Delta_F = -1/96\pi^2 a^2. \tag{43}$$

From (14) we therefore have

$$\bar{v}^2 = v^2 + (\hbar/32\pi^2 a^2). \tag{44}$$

Because the twisted field is antiperiodic in θ, the lowest eigenvalue of $-\Box$ is $l_0^2 = 1/4a^2$. The critical radius which follows from (17) is

$$a_c^2 = (4\lambda v^2)^{-1}(1 - \hbar\lambda/8\pi^2). \tag{45}$$

For $a < a_c$, $\varphi_q = 0$ is locally stable. For $a > a_c$, $\varphi_q = 0$ is unstable and symmetry breaking occurs. Note that one-loop quantum corrections lead to a decrease in the critical radius from its classical value.

Write $a = (1+\varepsilon)a_c$ as in (18). Then,

$$\begin{aligned}\Box &= \frac{\partial^2}{\partial t^2} + \frac{1}{a^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ &= \Box_c - \frac{2\varepsilon}{a_c^2}\frac{\partial^2}{\partial\theta^2} + \frac{3\varepsilon^2}{a_c^2}\frac{\partial^2}{\partial\theta^2} + O(\varepsilon^3),\end{aligned} \tag{46}$$

$$\bar{v}^2 = \bar{v}_c^2 - \frac{\varepsilon\hbar}{16\pi^2 a_c^2} + \frac{3\varepsilon^2\hbar}{32\pi^2 a_c^2} + O(\varepsilon^3). \tag{47}$$

We may read off the quantities $\bar{v}_n$ and $\Box_n$ by comparison with (19*b*, *c*).

The expression for $\tilde{\varphi}_0(x)$ is the eigenfunction of $-\Box_c$ with the lowest eigenvalue of $(4a_c^2)^{-1}$. It must therefore be static and independent of the y and z coordinates. We may choose

$$\tilde{\varphi}_0(x) = \cos\tfrac{1}{2}(\theta + \theta_0) \tag{48}$$

where θ_0 is an arbitrary constant reflecting the rotational invariance of the situation.

The expressions for A and B may be evaluated from (25), (26) to give

$$A = -2\pi\lambda v^2\varepsilon^2 V, \tag{49a}$$

$$B = \tfrac{3}{4}\pi\lambda\varepsilon^2 V, \tag{49b}$$

where $V = a\int \mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z$. The lowest-order contribution to the ground state from (30), (31) is

$$\varphi_q(x) = (\tfrac{8}{3}\varepsilon)^{1/2} v \cos\tfrac{1}{2}(\theta+\theta_0) \tag{50a}$$

$$= (\tfrac{8}{3})^{1/2} v (a/a_c - 1)^{1/2} \cos\tfrac{1}{2}(\theta+\theta_0). \tag{50b}$$

Note that the only role played by the quantum correction to this order is to change the critical radius.

In order to obtain the next-order contribution to φ_q as given in (41), we require a particular solution to (22) where

$$\varphi_0(x) = (\tfrac{8}{3})^{1/2} v \cos\tfrac{1}{2}(\theta+\theta_0) \tag{51}$$

appears in the inhomogeneous term on the right-hand side. It may be seen that

$$\psi_1(x) = -\tfrac{1}{3}(\tfrac{8}{3})^{1/2} v(1 - \hbar\lambda/8\pi^2)\cos^3\tfrac{1}{2}(\theta+\theta_0) \tag{52}$$

solves (22). The constants C and D appearing in (38), (39) may be evaluated to give

$$C = \tfrac{11}{6}(\tfrac{8}{3})^{1/2}\pi\lambda v^3(1 + 7\hbar\lambda/88\pi^2)V, \tag{53}$$

$$D = 4\pi\lambda v^2 V, \tag{54}$$

with $V = a\int \mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z$ as before. From (41) we have

$$\varphi_q(x) = (\tfrac{8}{3}\varepsilon)^{1/2} v \cos\tfrac{1}{2}(\theta+\theta_0)\{1 - \varepsilon[\tfrac{11}{24}(1 + 7\hbar\lambda/88\pi^2) + \tfrac{1}{3}(1 - \hbar\lambda/8\pi^2)\cos^2\tfrac{1}{2}(\theta+\theta_0)] + O(\varepsilon^2)\} \tag{55}$$

as the approximate ground state to this order. This result may be shown to agree with the first two terms in the expansion of the exact result of Avis and Isham (1978) (making the replacement $v^2 \to \bar{v}^2$). Note that in addition to changing the critical length, quantum corrections also alter the form of the $O(\varepsilon^{3/2})$ contribution to φ_q. Finally, we wish to remark that we could have used the rotational invariance to set $\theta_0 = 0$ in (48) if we had wished. It would then not have appeared in (55) although the solution above could be regained by invoking the rotational invariance.

3.2. $\mathbb{R}^1 \times \mathbb{RP}^3$

Consider the space–time whose line element is

$$\mathrm{d}s^2 = \mathrm{d}t^2 + a^2[\mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)]. \tag{56}$$

The usual topology is taken to be $\mathbb{R}^1 \times S^3$; however, as this is simply connected, no twisted real scalar fields are allowed. Instead, we may identify antipodal points on S^3 to give the spatial section the topology $S^3/Z_2 \approx \mathbb{RP}^3$, the real projective three-space. Since $\pi_1(\mathbb{R}^1 \times \mathbb{RP}^3) \approx Z_2$, twisted real scalar fields now exist. If $\boldsymbol{x}$ represents a point on S^3, which may be regarded as the surface $\boldsymbol{x}\cdot\boldsymbol{x} = 1$ imbedded in $\mathbb{R}^4$, then $-\boldsymbol{x}$ gives the point which is antipodal to $\boldsymbol{x}$. The twisted scalar field satisfies the boundary condition $\varphi(t, -\boldsymbol{x}) = -\varphi(t, \boldsymbol{x})$. In terms of the polar coordinates used in (56) we may write $\boldsymbol{x} = (x_1, \ldots, x_4)$:

$$x_1 = \sin\theta \cos\varphi \sin\chi, \tag{57a}$$

$$x_2 = \sin\theta \sin\varphi \sin\chi, \tag{57b}$$

$$x_3 = \cos\theta \sin\chi, \tag{57c}$$

$$x_4 = \cos\chi. \tag{57d}$$

The antipodal identification is then $(t, \chi, \theta, \varphi) \sim (t, \pi - \chi, \pi - \theta, \pi + \varphi)$.

Massless $\lambda\varphi^4$ theory has been considered previously in this space–time in Toms (1982a). Unwin (1982) has used Banach's (1981) method to study the twisted Goldstone model at the classical level in $\mathbb{R}^1 \times \mathbb{RP}^3$. Here we use the method described in § 2 to find the two lowest-order contributions to the ground state, including the effects of one-loop quantum corrections. The lowest-order classical limit of the resulting solution agrees with Unwin's (1982) result.

For simplicity we shall consider only the conformally coupled theory ($\xi = \frac{1}{6}$). The coincidence limit of the Feynman propagator defined in (10) is (Toms 1982a):

$$\Delta_{\mathrm{F}} = -1/12\pi^2 a^2. \tag{58}$$

From (14),

$$\bar{v}^2 = v^2 + \hbar/4\pi^2 a^2. \tag{59}$$

The d'Alembertian may be written as

$$\Box = \partial^2/\partial t^2 - \Delta_3/a^2 \tag{60}$$

where Δ_3 is the Laplacian on the unit three-sphere. The eigenfunctions of Δ_3 are harmonic polynomials in x_i (see (57)) which are homogeneous of degree n, $n = 0, 1, 2, \ldots$. The associated eigenvalues of Δ_3 are $n(n+2)$. (See Erdélyi *et al* (1953) p 232.) From the boundary condition satisfied by the twisted field, it follows that the allowed eigenfunctions of Δ_3 must have an odd degree. The lowest eigenvalue of $-\Box$ is therefore $l_0^2 = 3/a^2$. From (17) (with $\xi = \frac{1}{6}$, $R = 6/a^2$) the critical radius is found to be

$$a_{\mathrm{c}}^2 = (4/\lambda v^2)(1 - \hbar\lambda/16\pi^2). \tag{61}$$

For $a < a_{\mathrm{c}}$, $\varphi_q = 0$ is locally stable; for $a > a_{\mathrm{c}}$, $\varphi_q = 0$ is unstable and symmetry breaking occurs. One-loop quantum corrections again decrease the critical radius, although this is not a general feature.

The expression for $\tilde{\varphi}_0(x)$, which is the eigenfunction of $-\Box_{\mathrm{c}}$ with the lowest eigenvalue, is seen from (60) to be a polynomial in $(x_1, \ldots, x_4)$ homogeneous of degree one. There are clearly four possible choices as given in (57); however as Unwin (1982) has discussed, they are all related by an action of the isometry group (in this case SO(4) which acts transitively) so that we are free to choose whichever solution we like. This is analogous to the previous example where we could take $\theta_0 = 0$ in (48) by making use of the rotational symmetry. We choose

$$\tilde{\varphi}_0(x) = \cos\chi. \tag{62}$$

By expanding $\Box$, R, $\bar{v}^2$ using $a = (1+\varepsilon)a_{\mathrm{c}}$, we may read off

$$\Box_1 = 2\Delta_3/a_{\mathrm{c}}^2, \qquad \Box_2 = -3\Delta_3/a_{\mathrm{c}}^2, \tag{63a, b}$$

$$R_1 = -12/a_c^2, \qquad R_2 = 18/a_c^2, \tag{64a, b}$$

$$\bar{v}_1 = -\hbar/2\pi^2 a_c^2, \qquad \bar{v}_2 = 3\hbar/4\pi^2 a_c^2. \tag{65a, b}$$

It is now straightforward to evaluate the constants A and B given in (25), (26):

$$A = -\tfrac{1}{2}\pi^2 \lambda \varepsilon^2 a_c^3 v^2, \tag{66}$$

$$B = \tfrac{1}{8}\pi^2 \lambda \varepsilon^2 a_c^3. \tag{67}$$

Since $A < 0$, the solution (28) for $\hat{\varphi}$ makes the effective action a minimum. From (30) the lowest-order contribution to φ_q is

$$\varphi_q(x) = 2v\varepsilon^{1/2} \cos \chi. \tag{68}$$

If $\varphi_0(x) = 2v \cos \chi$ is substituted into the inhomogeneous term in (22), then a particular solution is found to be

$$\psi_1(x) = -\tfrac{2}{3}\lambda v^3 a_c^2 \cos^3 \chi. \tag{69}$$

The constants C and D which enter into the solution (41) are

$$C = -\tfrac{1}{3}\pi^2 \lambda v^3 a_c^3 (1 - 11\hbar\lambda/32\pi^2), \tag{70}$$

$$D = \pi^2 \lambda v^2 a_c^3. \tag{71}$$

Since $D > 0$, from (37) it is clear that $\hat{\varphi} = -C/D$ makes the effective action a minimum. The approximate ground state to order $\varepsilon^{3/2}$ is therefore

$$\varphi_q(x) = \varepsilon^{1/2}\{2v \cos \chi + \varepsilon[\tfrac{1}{3}v(1 - 11\hbar\lambda/32\pi^2) \cos \chi - \tfrac{2}{3}\lambda v^3 a_c^2 \cos^3 \chi] + O(\varepsilon^2)\}. \tag{72}$$

In general, if we had chosen any linear combination of the (x_1, x_2, x_3, x_4) in (57) for $\bar{\varphi}_0$ in (62), it is easy to see that we would end up with a solution for φ_q obtained from (72) by the replacement of $\cos \chi$ with $\Sigma_{i=1}^4 \alpha_i x_i$ where α_i are constants which satisfy $\Sigma_{i=1}^4 \alpha_i^2 = 1$.

4. Conclusions

In the preceding sections we have presented a method of finding vacuum solutions for twisted fields which are approximations to the true ground state, valid when the length scale characterising a phase transition is close to the critical value at which symmetry breaking occurs. Applications to two examples were given. In one of the examples where an exact solution is known, our approach gave perfect agreement with the first two terms in the expansion of this solution about the critical length.

The method presented in the present paper is not confined to twisted scalar fields and is applicable to any situation where symmetry breaking leads to a ground state which is not a constant. One such case of interest is field theory confined in a spatial cavity.

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Appendix 1

Consider a group G which acts on a manifold M in such a way that if $\varphi(x)$ is a solution to the field equations, so is $\varphi(gx)$ for $g \in \mathrm{G}$, $x \in M$. Assume that $\varphi(gx)$ is not identical to $\varphi(x)$ for all g, x, but that

$$\Gamma[\varphi(x)] = \Gamma[\varphi(gx)] \tag{A1.1}$$

where Γ is the effective action. (In the example in § 3.1 if $\varphi(x) = \cos\frac{1}{2}\theta$, then $\varphi(gx) = \cos\frac{1}{2}(\theta+\theta_0)$. In § 3.2 we may take $\mathrm{G} = \mathrm{SO}(4)$.) Write

$$gx = x + \delta x, \tag{A1.2}$$

$$\varphi(gx) = \varphi(x) + \delta\varphi(x). \tag{A1.3}$$

Expand the effective action in a functional Taylor series about a solution $\varphi_q(x)$:

$$\Gamma[\varphi_q(gx)] = \Gamma[\varphi_q(x)] + \tfrac{1}{2}\int \mathrm{d}v_x\, \mathrm{d}v_{x'}\, \delta\varphi(x) \frac{\delta^2\Gamma[\varphi]}{\delta\varphi(x)\delta\varphi(x')}\bigg|_{\varphi_q} \delta\varphi(x') + \dots \tag{A1.4}$$

Because of (A1.1), and our assumption that $\delta\varphi \neq 0$, this shows that

$$\delta^2\Gamma[\varphi]/\delta\varphi(x)\delta\varphi(x')|_{\varphi_q} \tag{A1.5}$$

must have a zero mode. If the solution φ_q is not to be unstable, then λ_0, the lowest eigenvalue of (A1.5), must be zero. This argument is a generalisation of one in Rajaraman (1975).

The equation whose solution determines the stability of φ_q was given in (16) as

$$[-\Box_x + \xi R + \lambda(3\varphi_q^2(x) - \bar{v}^2)]\Psi_N(x) = \lambda_N \Psi_N(x). \tag{A1.6}$$

Define

$$H_0 = -\Box_x + \xi R - \lambda\bar{v}^2, \tag{A1.7}$$

$$H_I = 3\lambda\varphi_q^2(x), \tag{A1.8}$$

with

$$H_0\Psi_N^{(0)}(x) = \lambda_N^{(0)}\Psi_N^{(0)}(x). \tag{A1.9}$$

We know that if $\varphi_q = 0$ is unstable for $L = (1+\varepsilon)L_c$ then $\lambda_0^{(0)} < 0$ for $\varepsilon > 0$. Also, by definition of L_c in (17), we have $\lambda_0^{(0)} = 0$ for $\varepsilon = 0$. Expand H_0 in powers of ε using (19a, b, c):

$$H_0 = h_0 + \varepsilon h_1 + \mathrm{O}(\varepsilon^2). \tag{A1.10}$$

Because $\lambda_0^{(0)} = 0$ for $\varepsilon = 0$, we must have $h_0\Psi_0^{(0)} = 0$; thus,

$$\lambda_0^{(0)} = \varepsilon \int \bar{\Psi}_0^{(0)} h_1 \Psi_0^{(0)} + \mathrm{O}(\varepsilon^2). \tag{A1.11}$$

It therefore follows that $\lambda_0^{(0)}$ is negative and of order ε. We then add in H_I, considered as a perturbation; from the argument above we must have

$$\lambda_0^{(0)} + \lambda_0^{(1)} = 0 \tag{A1.12}$$

to leading order in ε, where

$$\lambda_0^{(1)} = \int \bar{\Psi}_0^{(0)} H_I \Psi_0^{(0)}. \tag{A1.13}$$

Since H_I involves φ_q^2 we conclude that $\varphi_q = O(\varepsilon^{1/2})$. In fact this argument may be used to determine the coefficient $\hat{\varphi}$ of $\tilde{\varphi}_0$ in (23) to give a result in agreement with that of § 2 above. By using second-order perturbation theory, the coefficient of the term of order $\varepsilon^{3/2}$ in φ_q may also be obtained in this way.

Finally, we wish to remark that a more general argument for the behaviour near the critical length can be given based on a renormalisation group analysis (see Amit 1978 and references therein).

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